CAUCHY'S THEOREM

If a function $f(z)$ is analytic and $f'(z)$ is continuous at each point within and on a simple closed contour γ, then

$$\oint_\gamma f(z)\,dz = 0$$

Proof: We know that

$$\oint_\gamma f(z)dz = \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (v\,dx + u\,dy)$$

Since $f'(z)$ is continuous, the partial derivatives of u and v are continuous, and Green's Theorem applies:

$$\oint_\gamma f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

Since $f(z)$ is analytic, the Cauchy-Riemann equations hold, so that

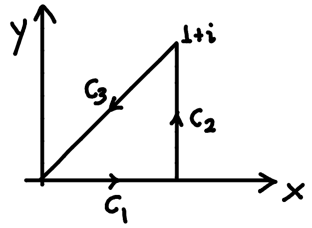
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and therefore

$$\oint_\gamma f(z)dz = 0 .$$

Remarks:

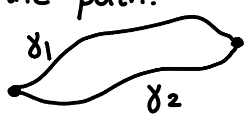
- This theorem predicts the result of the previous example:



$$\oint_C z^2\,dz = 0, \quad \text{where} \quad C = C_1 + C_2 + C_3$$

as z^2 is analytic.

- It implies that the definite integral of an analytic function is independent of the path:



Along the closed curve $\gamma = \gamma_1 - \gamma_2$,

$$\oint_{\gamma_1 - \gamma_2} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz - \oint_{\gamma_2} f(z)\,dz = 0, \quad \text{so} \quad \oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$$

- Furthermore, if $f(z)$ is continuous and $\oint f(z)\,dz = 0$ for every simple closed contour γ in a domain D, we can integrate directly with respect to z.

e.g.

$$\int_{i+1}^{0} z^2\,dz = \frac{1}{3}\left[z^3\right]_{i+1}^{0} = \frac{-1}{3}(i+1)^3 = \frac{2}{3} - \frac{2i}{3},$$

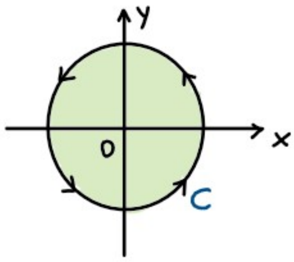
as found in the final step of that example.

- In the example: $\oint_C \bar{z}\,dz = 2\pi i R^2$, where C is a circular contour of radius R, $\oint_C \bar{z}\,dz$ is non-zero because $f(z) = \bar{z}$ is nowhere analytic.

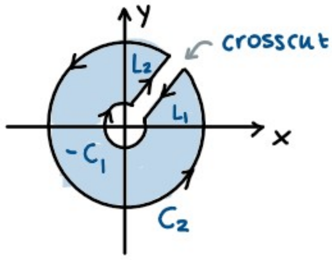
Deformation of the contour

In the example: $\int_C z^{-1} dz = 2\pi i$, where C is a circular contour of radius R,

the integral is non-zero again, since $f(z) = \frac{1}{z}$ is not analytic at each point within C. The origin must be excluded.



Original Contour C
(includes the origin)



New contour (excludes the origin)
$C' = C_2 + L_1 + L_2 - C_1$

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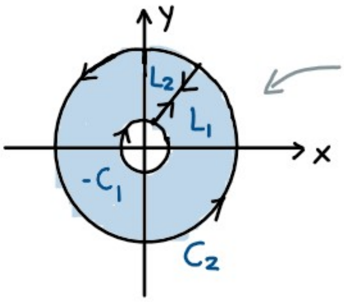
The function $f(z) = \frac{1}{z}$ is now analytic everywhere within C'.
So, by Cauchy's Theorem:

$$\oint_{C'} \frac{1}{z} dz = 0 \quad , \quad \text{or}$$

$$\oint_{C_2} z^{-1} dz + \int_{L_1} z^{-1} dz + \int_{L_2} z^{-1} dz - \oint_{C_1} z^{-1} dz = 0 .$$



As the width of the crosscut $\to 0$, the integrals on L_1 and L_2 cancel because $L_2 \to -L_1$. So the above relation implies that

$$\oint_{C_2} z^{-1} dz = \oint_{C_1} z^{-1} dz$$

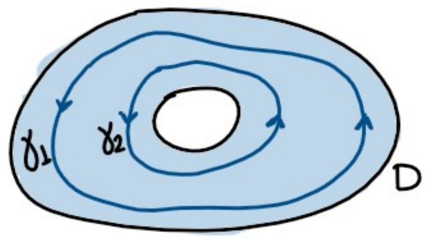
although C_1 and C_2 have different radius.

It was found (by contour integration along $\gamma(t) = e^{it}$) that

$$\oint_{C_1} z^{-1} dz = 2\pi i$$

The concept of deformation

Functions, like $\frac{1}{z}$, that are not entirely analytic on the domain enclosed by γ cannot follow Cauchy's Theorem. However, we avoided the origin by following the contour C' and shrinking its crosscut. This strategy, when generalised, gives the following result:



Let f be analytic on D. D has a hole and therefore does not contain the interior of the curves γ_1, γ_2. The curves γ_1, γ_2 are supposed to be simple and closed. Then:

$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz$$

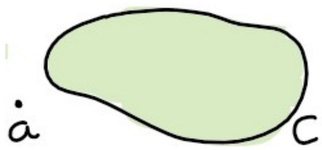
implying that the integral is the same along any simple closed (anti-clockwise) curve γ enclosing the hole.

This process is called deformation because γ_1 can be deformed to γ_2 without affecting the integral.

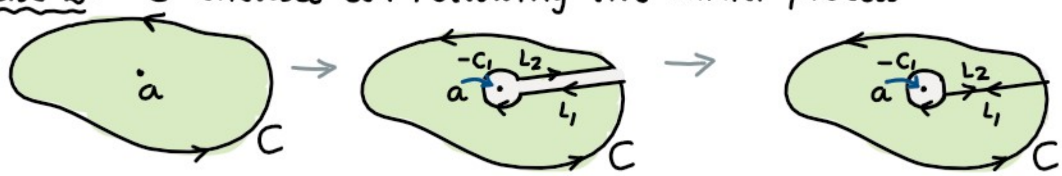
EXAMPLE Evaluate $I = \oint_C \frac{dz}{(z-a)^m}$ for $m = 1, 2, 3, \ldots, M$

where C is a simple closed contour.

Case 1: C does not enclose a. Since $f(z)$ is analytic $\forall z \neq a$, we have that $I = 0$ by Cauchy's theorem.



Case 2: C encloses a. Following the earlier process



$$\oint_C f + \cancel{\int_{L_1} f} + \cancel{\int_{L_2} f} - \oint_{C_1} f = 0, \text{ by Cauchy's Theorem}$$

and so

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.$$

To evaluate $\oint_{C_1} f(z)\,dz$, we let $z - a = re^{i\theta}$, $dz = ire^{i\theta}d\theta$.

Then, $I = \oint_{C_1} \frac{1}{(z-a)^m}dz = \int_0^{2\pi} \frac{1}{r^m e^{im\theta}} ire^{i\theta}d\theta = i\int_0^{2\pi} r^{-m+1} \exp(-i(m-1)\theta)\,d\theta.$

- When $m = 1$, $I = i\int_0^{2\pi} e^0 r^0 d\theta = i\int_0^{2\pi} d\theta = 2\pi i$
- when $m \neq 1$, $I = \left[\frac{-i\exp(-i(m-1)\theta)\,r^{-m+1}}{i(m-1)}\right]_0^{2\pi} = \frac{-r^{-m+1}}{m-1}\left[1-1\right] = 0$

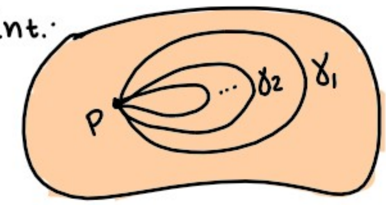
Remark: I is non-zero only when $m = 1$. We will focus on this result later.

Cauchy's Integral Formula

Cauchy's Theorem yields Cauchy's integral formula, which we discuss here.

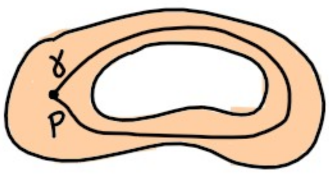
➔ The domain with no "holes" is called simply connected:

➔ Any simple closed curve that does not enclose a "hole" we say that is homotopic to a point. Formally, we say that it can be continuously deformed to a point:



$\gamma_n \xrightarrow[n\to\infty]{} P$

If γ encloses a hole, cannot shrink to a point:



and is <u>not</u> homotopic to a point.

Cauchy's Integral Formula

Let f be analytic on a region D enclosed by a simple closed curve γ. Then, at any point z_0 in D,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

<u>Remark</u>: The conditions of the theorem imply that D is a simply connected region and that γ is homotopic to a point.

Proof

Consider a small circle γ_ε with radius ε and centre at z_0.
Since $\frac{f(z)}{z-z_0}$ is analytic for $z \neq z_0$, we can deform γ into γ_ε and

$$\oint_\gamma \frac{f(z)}{z-z_0}\,dz = \oint_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,dz$$

An equation for γ_ε is $|z-z_0| = \varepsilon$ or

$$z - z_0 = \varepsilon e^{i\varphi}, \text{ with } 0 \leq \varphi < 2\pi.$$

We then have $z = z_0 + \varepsilon e^{i\varphi}$ and $dz = i\varepsilon e^{i\varphi}\,d\varphi$,

so that
$$\oint_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{i\varphi})\, i\varepsilon \cancel{e^{i\varphi}}}{\cancel{\varepsilon e^{i\varphi}}}\,d\varphi = i\int_0^{2\pi} f(z_0+\varepsilon e^{i\varphi})\,d\varphi.$$

Since the two integrals $\oint_\gamma \frac{f}{\Delta z}$, $\oint_{\gamma_\varepsilon} \frac{f}{\Delta z}$ are equal,

$$\oint_\gamma \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0+\varepsilon e^{i\varphi})\,d\varphi.$$

The radius ε of the small circle can be as small as we like, so

$$\oint_\gamma \frac{f(z)}{z-z_0}\,dz = i\cdot \lim_{\varepsilon\to 0}\int_0^{2\pi} f(z_0+\varepsilon e^{i\varphi})\,d\varphi = i\int_0^{2\pi} \lim_{\varepsilon\to 0} f(z_0+\varepsilon e^{i\varphi})\,d\varphi$$

$$= i\int_0^{2\pi} f(z_0)\,d\varphi \quad \text{(since } f \text{ is continuous)}$$

$$= i f(z_0)\int_0^{2\pi} d\varphi = \underline{2\pi i f(z_0)}$$

Then,

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\,dz \quad \text{as required.}$$

- The theorem means that the values of an analytic function inside a simple closed curve are entirely fixed by its values on the boundary (the points of the curve.)

- Cauchy's integral formula can replace integration by evaluation of a function (see the examples below).

EXAMPLE Evaluate $\oint_C \frac{5z-2}{z-1}\,dz$, where C is the Circle $|z|=2$.

- Here, $f(z)=5z-2$, which is analytic on $\mathbb{C}$.
- Also, $z=1$ lies within the contour, so we can use Cauchy's integral formula:

$$\oint \frac{f(z)}{z-1}\,dz = 2\pi i f(1) = 2\pi i\,(5\cdot 1-2) = 6\pi i \;.$$

EXAMPLE Evaluate $\oint_C \frac{e^z}{z^2-5z+6}\,dz$, where C is the Circle $|z|=5$.

The partial fraction form of $\frac{1}{z^2-5z+6}$ is $\frac{1}{z-3} - \frac{1}{z-2}$

($z_1=3$, $z_2=2$ are the roots of z^2-5z+6 and then we solve $\frac{1}{z^2-5z+6} = \frac{A}{z-3} + \frac{B}{z-2}$ with respect to A and B)

Also $f(z)=e^z$. So,

$$\oint_C \frac{e^z}{z^2-5z+6}\,dz = \oint_C \frac{e^z}{z-3}\,dz - \oint_C \frac{e^z}{z-2}\,dz = 2\pi i f(3) - 2\pi i f(2)$$

$$= 2\pi i\left(e^3 - e^2\right) .$$

Remark: Notice that in all of the above examples the value of the radius R of the circle $|z|=R$ does not enter in the integral evaluation process.

Cauchy's Integral Formula for derivatives

We can differentiate Cauchy's integral formula with respect to z_0 to find

$$f'(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^2}\,dz$$

and

$$f''(z_0) = \frac{2}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^3}\,dz .$$

Suppose that

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Differentiating $f^{(k)}(z_0)$ we find

$$f^{(k+1)}(z_0) = \frac{(k+1)!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+2}}\,dz$$

and by induction

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

EXAMPLE Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is the circle $|z|=3$.

Let $f(z)=e^{2z}$, $z_0=-1$ and $k=3$ in Cauchy's integral formula for the derivatives:

$$f^{(3)}(-1)=\frac{3!}{2\pi i}\oint_\gamma \frac{f(z)}{(z+1)^4}\,dz=\frac{3!}{2\pi i}\oint_\gamma \frac{e^{2z}}{(z+1)^4}\,dz$$

or

$$\oint_\gamma \frac{e^{2z}}{(z+1)^4}\,dz=\frac{2\pi i}{3!}f^{(3)}(-1).$$ So, we only need to find $f^{(3)}(-1)$.

It is: $f'(z)=2e^{2z}$, $f''(z)=4e^{2z}$ and $f^{(3)}(z)=8e^{2z}$.

So, $f^{(3)}(-1)=8e^{-2}$, and

$$\oint_\gamma \frac{e^{2z}}{(z+1)^4}\,dz=\frac{2\pi i}{3!}8e^{-2}=\frac{8}{3}\pi i e^{-2}.$$

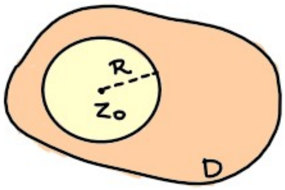
Cauchy's Estimate or Cauchy's Inequalities

These inequalities give an upper bound on the magnitude of derivatives.

Given Cauchy's integral formula for the derivatives

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz \qquad (1)$$

where γ is the circle $|z-z_0| = R$, we suppose that $f: D \to \mathbb{C}$ is bounded for all z on γ, i.e.



$$|f(z)| < M \quad , \quad \forall z \text{ such that } |z-z_0| = R.$$

It turns out that all derivatives of f at z_0 are also bounded and in particular:

$$|f^{(k)}(z_0)| \leq \frac{k!M}{R^k} \quad , \quad \text{for any } k=0,1,2,\dots$$

From (1) we have

$$|f^{(k)}(z_0)| \leq \frac{k!}{2\pi} \oint_\gamma \frac{|f(z)|}{|z-z_0|^{k+1}} |dz|$$

$$\leq \frac{k!\, M}{2\pi R^{k+1}} \oint_\gamma |dz|$$

$$\leq \frac{k!\, M}{2\pi R^{k+1}} \cdot 2\pi R = \frac{k!M}{R^k}$$