LAPLACE'S EQUATION AND HARMONIC FUNCTIONS

Definition Let $u: D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$ and $u(x,y)$ is twice differentiable. The function u is called harmonic function if

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\nabla^2 u$ is called the Laplacian of u and the above equation Laplace's equation.

- Laplace's equation rises naturally in a wide range of problems, e.g. heat conduction, fluid flow.

PROPOSITION Let $f: D \to \mathbb{C}$ be an analytic complex function on an open subset D of $\mathbb{C}$ and $f = u + iv$ ($u = \mathrm{Re} f$, $v = \mathrm{Im} f$). Then u and v are harmonic.

Proof: Since f is analytic, u, v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We differentiate them with respect to x and y accordingly:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

It holds that $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$, so by adding the above equations, we find:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{i.e. } u \text{ is harmonic.}$$

With a similar argument we find that v is also harmonic ($v_{xx} + v_{yy} = 0$).

→ The real and the imaginary parts of $f = u + iv$ in the previous proposition are called harmonic conjugates.

→ $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ from the last example are harmonic conjugates of each other according to the proposition, since $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, where f is the analytic function:

$$f(z) = z^2.$$

Remark: Cauchy-Riemann equations imply that an analytic function can be determined by its real part only (or its imaginary part only).

Remark: We can easily find pairs of real two-variable functions $g_1(x,y)$, $g_2(x,y)$ which are harmonic conjugates of each other, from the real and the imaginary parts of analytic complex functions, e.g. $f(z) = z^3$, $z^2 - 3z + 1$, $\sin z$, etc.

→ The Cauchy-Riemann Theorem implies that the derivative of an analytic function at z_0 is:

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$ (see the proof of C-R theorem for $x \to x_0$)

but also

$$f'(z_0) = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$$ (proof of C-R theorem for $y \to y_0$)

Cauchy-Riemann in polar coordinates

To convert the Cauchy-Riemann equations to polar form, we use the chain rule for $u(x,y)$ and $v(x,y)$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

(same for $v(x,y)$)

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and therefore:

$$\left.\begin{aligned} u_r = \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\ u_\theta = \frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta \end{aligned}\right\} (*)$$

and similarly,

$$\left\{\begin{aligned} v_r = \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\ v_\theta = \frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta \end{aligned}\right.$$

We will express v_r, v_θ in terms of u_r, u_θ by using the C-R equations $u_x = v_y$, $v_x = -u_y$:

$$\left\{\begin{aligned} v_r &= -u_y\cos\theta + u_x\sin\theta = u_x\sin\theta - u_y\cos\theta \overset{(*)}{=} -\frac{1}{r}u_\theta \\ v_\theta &= u_y r\sin\theta + u_x r\cos\theta = r(u_x\cos\theta + u_y\sin\theta) \overset{(*)}{=} r\,u_r \end{aligned}\right.$$

and in short:
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

DIFFERENTIATION OF BASIC COMPLEX FUNCTIONS

We expand the discussion on differentiation of complex functions.
We start by the exponential function:

$$f: \mathbb{C} \longrightarrow \mathbb{C}$$
$$f(z) = e^z$$

To determine that e^z is analytic, we show that the real and imaginary parts are harmonic conjugates.

Simply, $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

so $u(x,y) = e^x \cos y = \mathrm{Re}(e^z)$

and $v(x,y) = e^x \sin y = \mathrm{Im}(e^z)$.

To verify the Cauchy-Riemann equations, we first find the partial derivatives:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= (e^x)' \cos y = e^x \cos y \\ \frac{\partial v}{\partial y} &= e^x (\sin y)' = e^x \cos y \end{aligned}\right\} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= e^x (\cos y)' = -e^x \sin y \\ \frac{\partial v}{\partial x} &= (e^x)' \sin y = e^x \sin y \end{aligned}\right\} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So e^z is analytic and its derivative is $\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ (Cauchy-Riemann Theorem's proof)

$$\frac{df}{dz} = e^x \cos y + i e^x \sin y = e^z.$$

This is the full proof why $\frac{de^z}{dz} = e^z$. Based on this and on differentiation rules we differentiate combinations of functions:

$$\frac{d(e^{2z} + z + 1)}{dz} = 2e^{2z} + 1, \quad \frac{d(e^{z^4})}{dz} = 4z^3 e^{z^4}, \quad \text{etc.}$$

The logarithm [see also the last 2 pages of notes 1]

The logarithm is an inverse of $z = e^w$:

$$w = \ln z$$

not defined on the whole plane $\mathbb{C}$ as e^w but on $\mathbb{C} \setminus \{0\}$.

$$f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$$
$$f(z) = \ln z$$

We can re-write $z = e^w$ in its polar form

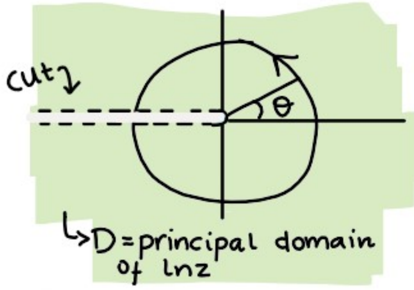
$$z = r e^{i\theta + 2\pi k i} \quad \text{for } 0 \leq \theta < 2\pi, \ k \in \mathbb{Z}.$$

to express the logarithm in polar coordinates

$$w = \ln z = \ln r + i\theta + 2\pi k i$$

The logarithm can take an infinite number of values, since k can take an an infinite number of values. Thus the logarithm is a multi-valued function.

- It is often useful to restrict multi-valued functions so that they become single-valued.
- For $k=0$, we have the principal value of the logarithm.
- To restrict $\ln z$ and make it a single-valued function, we need the concept of a branch point: if we perform a complete circuit around a closed path in $\mathbb{C}$, that includes a branch point of $f(z) = \ln z$.



after a closed circuit around 0, θ is increased by 2π.

- $\arg z$ is a discontinuous function, so
- $\ln z = \ln|z| + i \arg z$ cannot be differentiable on its whole domain $\mathbb{C} \setminus \{0\}$.

- $\ln z$ is analytic and $\dfrac{d \ln z}{dz} = \dfrac{1}{z}$ on D = restricted domain on which there is a single-valued branch of $\ln z$.

- Principal branch: for $0 < \theta < 2\pi$ or $-\pi < \theta < \pi$

  For $-\pi < \theta < \pi$: $D = \mathbb{C} \setminus \{x + iy : x \leq 0 \text{ and } y = 0\}$

$\ln z$ is analytic and $\frac{d\ln z}{dz} = \frac{1}{z}$

We show that the restricted logarithm (single-valued for $k=0$) is analytic.

$$\ln z = \underbrace{\ln r}_{u(r,\theta)} + i\underbrace{\theta}_{v(r,\theta)} \qquad \text{(in polar coordinates)}$$

The Cauchy-Riemann equations in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

are satisfied since $u_r = \frac{1}{r}$, $v_\theta = 1$ and $u_\theta = v_r = 0$.
Then,

$$\frac{d\ln z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + i\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{1}{r}\frac{\partial r}{\partial x} + i\frac{\partial \theta}{\partial x}$$

Since $\left.\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}\right\}$ we get $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

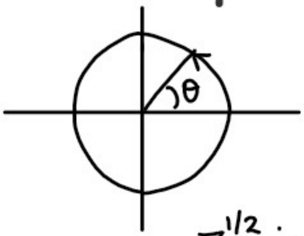
and $\frac{\partial \theta}{\partial x} = \cdots = -\frac{y}{r^2}$.

Hence, the derivative is

$$\frac{d\ln z}{dz} = \frac{x}{r^2} - i\frac{y}{r^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z}$$

(remember that $z\bar{z} = |z|^2$)

The square root function $f(z) = z^{1/2}$

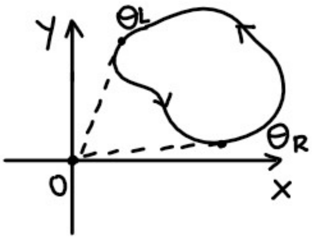


$f: \mathbb{C} \to \mathbb{C}$ is a multi-valued function
$$f(z) = z^{1/2} = r^{1/2} e^{i\left(\frac{\theta}{2} + k\pi\right)}, \; k \in \mathbb{Z}.$$

$z^{1/2}$ is defined as the inverse of z^2. So, let $z = w^2$ we will then solve it in terms of $w = z^{1/2}$:

Let $w = \rho e^{i\varphi}$ and $z = re^{i\theta}$, then $re^{i\theta} = \rho^2 e^{2i\varphi}$ (1) which yields that $|re^{i\theta}| = |\rho^2 e^{2i\varphi}|$ or $\rho = r^{1/2}$. From (1), $e^{i\theta} = e^{2i\varphi}$ or $\theta + 2k\pi = 2\varphi$, $k \in \mathbb{Z}$.
So, $\varphi = \frac{\theta}{2} + k\pi$, $k \in \mathbb{Z}$ and $w = z^{1/2} = \rho e^{i\varphi} = r^{1/2} e^{i\left(\frac{\theta}{2} + k\pi\right)}$, $k \in \mathbb{Z}$.

It also means that $f(z)$ changes from $r^{1/2} e^{\frac{i\theta}{2}}$, $k=0$ to $r^{1/2} e^{i\left(\frac{\theta}{2} + \pi\right)}$, $k=1$ which is $r^{1/2} e^{i\left(\frac{\theta}{2}+\pi\right)} = r^{1/2} e^{\frac{i\theta}{2}} \cdot e^{i\pi} = -r^{1/2} e^{\frac{i\theta}{2}} \to$ exactly the opposite of its original value ($k=0$).



- The origin is a branch point of $z^{1/2}$.
- However, if we loop around a closed curve that does not contain the origin, then θ lies in a restricted range $[\theta_L, \theta_R]$ and returns to its original value after one circuit.

- We can write $f(z) = z^{\frac{1}{2}} = e^{\frac{1}{2}\ln z}$, where we know that $\ln z$ is single-valued and analytic on its principal branch with domain $D = \mathbb{C} \setminus \{x+iy : x \le 0, \; y=0\}$.

→ Therefore, $z^{1/2}$ is analytic on D and its derivative is:

$$\frac{dz^{1/2}}{dz} = \frac{de^{\frac{1}{2}\ln z}}{dz} = \left(\frac{1}{2}\ln z\right)' \cdot e^{\frac{1}{2}\ln z} = \frac{1}{2}\frac{1}{z} z^{\frac{1}{2}} = \frac{1}{2} z^{-\frac{1}{2}}.$$

Remark: In general, z^a, $a \in \mathbb{R}$ is analytic on the domain
$$D = \mathbb{C} \setminus \{x+iy : x \le 0, \; y=0\}$$
and
$$\frac{dz^a}{dz} = a z^{a-1}.$$

Trigonometric Functions

The sine and cosine functions are defined on the whole complex plane.

We have that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

By the basic properties of differentiation :

$$\frac{d\sin z}{dz} = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{1}{2i}\left(\frac{de^{iz}}{dz} - \frac{de^{-iz}}{dz}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

and

$$\frac{d\cos z}{dz} = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{2}\left(\frac{de^{iz}}{dz} + \frac{de^{-iz}}{dz}\right) = i\frac{e^{iz} - e^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

Remark : The inverse of $\sin z$ and $\cos z$ are multi-valued functions.

We can calculate their inverses in the same way as for the real hyperbolic functions.

Let $z = \cos w = \dfrac{e^{iw} + e^{-iw}}{2}$. Then $2z = e^{iw} + e^{-iw}$

or $e^{2iw} - 2z e^{iw} + 1 = 0$

The roots of this quadratic expression in terms of e^{iw} are:

$$e^{iw} = \frac{2z \pm (4z^2 - 4)^{1/2}}{2} = z \pm (z^2 - 1)^{1/2}$$

or $$w(z) = \frac{1}{i}\ln\left(z \pm (z^2 - 1)^{1/2}\right).$$

However, $w(z)$ is multi-valued because it is composed by the logarithm and the square root function.

So, the inverse of the cosine function

$$\cos^{-1}(z) = \frac{1}{i}\ln\left(z \pm (z^2-1)^{1/2}\right)$$

is also a multi-valued function, which can become single-valued when restricted, for example, on its principal branch when $k=0$:

$$\cos^{-1}(z) = \frac{1}{i}\ln\left(z + (z^2-1)^{1/2}\right)$$

and in that domain, the negative root is excluded.

EXAMPLE Find a domain where $f(z) = \sqrt{z+1}$ is analytic.
(Also called the analyticity domain).

Since an analyticity domain for $\sqrt{w}$, $w \in \mathbb{C}$ is:

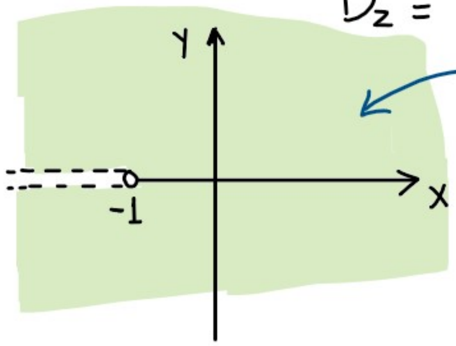
$$D_w = \mathbb{C} \setminus \{x+iy : x \leq 0,\ y=0\}$$

an analyticity domain for $\sqrt{z+1}$ is:

$$D_z = \mathbb{C} \setminus \{z \in \mathbb{C} : Re(z+1) \leq 0,\ Im(z+1) = 0\}.$$

Since, $z+1 = (x+1) + iy$, we have $Re(z+1) = x+1$ and $Im(z+1) = y$.
Hence,

$$D_z = \mathbb{C} \setminus \{x+iy : x \leq -1,\ y=0\}$$



the analyticity domain of $\sqrt{z+1}$.

COMPLEX INTEGRATION

One of the most important theorems in complex integration is the so-called Cauchy's Theorem. But before stating this theorem, we first define the contour integrals.

In the complex plane, we can integrate along curves. Let

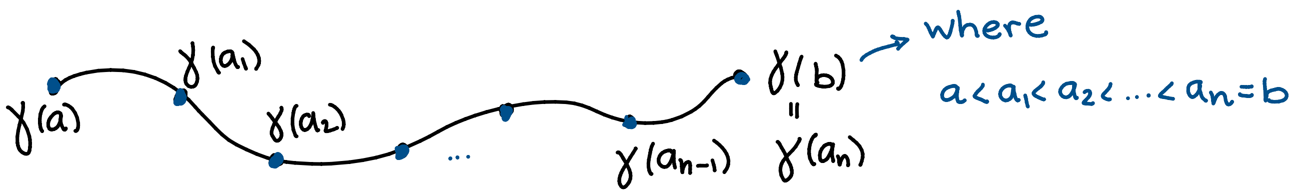
$$\gamma : [a,b] \subset \mathbb{R} \to \mathbb{C}$$
$$\gamma(t) = x(t) + i y(t)$$

be a smooth path in $\mathbb{C}$.

- On a smooth path $\gamma(t)$, $\gamma'(t)$ is continuous.
- Recall that: paths can be added $(\gamma_1 + \gamma_2)$, but also can be splitted.

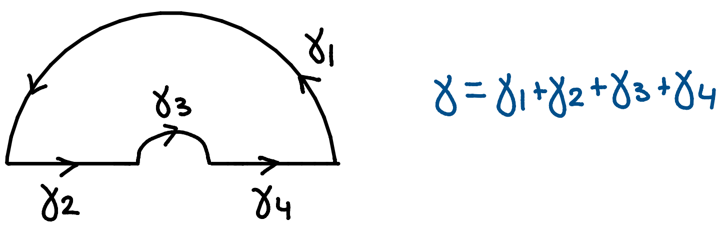
Definition A contour γ is a continuous path $\gamma : [a,b] \to \mathbb{C}$.
The path γ is called piecewise continuous if we can split the interval $[a,b]$ into a finite number of subintervals

$$a = a_0 < a_1 < \dots < a_n = b$$



such that $\gamma'(t)$ exists on each subinterval (a_i, a_{i+1}) and is continuous on $[a_i, a_{i+1}]$.

- A contour consists of a finite number of connected smooth curves.



The contour integral

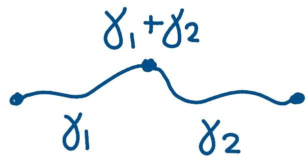
The integral along a piecewise contour γ of a continuous function $f(z)$ is defined to be

$$\int_{\gamma} f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt,$$

where we have converted the integral over z into an integral over t.

Properties of the contour integral

- $\int_C [\alpha f(z) + \beta g(z)]\,dz = \alpha\int_C f(z)\,dz + \beta\int_C g(z)\,dz$, α, β constants
- $-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$, where $-C$ denotes the opposite path.
- $\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$



- $\left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \leq \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt$

Definition The arc length of a curve $\gamma:[a,b]\to\mathbb{C}$, $\gamma(t) = x(t) + iy(t)$ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

- The arc length of the unit circle is equal to 2π. ($\gamma:[0,2\pi]\to\mathbb{C}$, $\gamma(t) = e^{it}$ in polar coordinates)