MTH2003M Complex Analysis **Practical 5**

1. For each sequence below, find $f(z)=\lim_{n\to\infty} f_n(z)$, and show that the sequence is uniformly convergent to $f(z)$ for $|z|\le 1$:

   (a) $f_n(z)=1+\dfrac{z}{n}$, $n=1,\ 2,\ldots$ (b) $f_n(z)=2z+\dfrac{3z^2}{n}$ for $n=1,\ 2,\ldots$

2. Given that the p-series $\sum_n \dfrac{1}{n^p}$ converges for $p>1$, show that the series

$$\sum_{n=1}^{\infty}\frac{z^{n+1}}{n^6\sqrt{2+n}},$$

   converges uniformly for $|z|<1$.

3. Show that the sequences below converge uniformly in the given regions ($\alpha\in\mathbb{R}$ is a constant):

   a) $f_n(z)=\dfrac{1}{nz^2},\ 0<\alpha\le|z|<\infty$ b) $f_n(z)=\dfrac{1}{z^n},\ 1<\alpha\le|z|<\infty$

   Would your answer to a) be different with the interval $0<|z|<\infty$?

4. Show that the following series converge uniformly in the given regions ($R\in\mathbb{R}$ is a constant):

   a) $\sum_{n=1}^{\infty} z^n,\ 0\le|z|<R,\ R<1$ b) $\sum_{n=1}^{\infty} e^{-nz},\ R<\operatorname{Re} z\le 1,\ R>0$

5. Find the radius of convergence of each of the following series:

   a) $\sum_{n=1}^{\infty}\dfrac{z^n}{n}$ b) $\sum_{n=1}^{\infty}\dfrac{n}{2^n}z^n$ c) $\sum_{n=1}^{\infty}(\cos in)z^n$