ALGEBRAIC STRUCTURES - PRACTICAL 8

This week's Exercises

Solve **all** the exercises of this section (i.e. Exercises 7.1 to 7.3) before the beginning of Semester B.

8.1. Compute the following principal ideals.

(*1*) $(\overline{3})$ in $\mathbb{Z}/9\mathbb{Z}$,
(*2*) $(\overline{3})$ in $\mathbb{Z}/7\mathbb{Z}$,
(*3*) $(\overline{4})$ in $\mathbb{Z}/12\mathbb{Z}$,
(*4*) $(\overline{4})$ in $\mathbb{Z}/16\mathbb{Z}$.

8.2. Compute the characteristic of the following rings:

(a) $\mathbb{F}_2 \times \mathbb{F}_3$ (the Cartesian product ring, also called direct product, of the fields $\mathbb{F}_2 = (\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{F}_3 = (\mathbb{Z}/3\mathbb{Z})$);
(b) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$;
(c) $\mathbb{C} \times \mathbb{F}_2$;
(d) $\mathbb{F}_{31}[x, y]$, the polynomial ring in two variables x and y and coefficients in the field $\mathbb{F}_{31} = \mathbb{Z}/31\mathbb{Z}$.

8.3. [Important properties of cosets]. Let I be an ideal of a ring R and $a, b \in R$.

(*1*) Show that $a + I = I$ if and only if $a \in I$.

(*2*) Show that either $a + I = b + I$ or $a + I \cap b + I = \varnothing$.
(That is, two cosets are either disjoint or the same.)

(*3*) Show that $a + I = b + I$ if and only if $a - b \in I$.

Practice Exercises

The next exercises are meant to give you some extra practice to better prepare for assessments.

8.4. Let R be a ring and let I be an ideal of R.
Verify that $\eta : R \to R/I$ given by

$$\eta(r) = r + I$$

is a ring homomorphism.

8.5. In the quotient ring $\mathbb{F}_3[x]/J$ where $J = (x^3 + 2)$, describe the following elements (which are cosets) in the form

$$(a + bx) + J.$$

(a) $(2 + x + 2x^2 + J)(1 + x^2 + J)$;
(b) $(1 + 2x + x^4 + J) + (x^3 + J)$.

8.6. Let I be an ideal of a ring R. Show that the quotient ring R/I is commutative if and only if $ab - ba \in I$ for all $a, b \in R$.

8.7. Consider polynomials with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Express your answers to the following questions in terms of n and d.

(a) In $(\mathbb{Z}/n\mathbb{Z})[x]$, how many different polynomials are there of degree $\leq d$?

(b) In $(\mathbb{Z}/n\mathbb{Z})[x]$, how many different monic polynomials are there of degree d?

(c) In $(\mathbb{Z}/n\mathbb{Z})[x]$, how many different polynomials are there of degree d?

8.8. Find the field of fractions of the following rings.

(*1*) The integers $\mathbb{Z}$,

(*2*) The polynomial ring $\mathbb{Q}[x]$,

(*3*) The finite field $\mathbb{F}_p$ (p prime),

(*4*) The ring of Gaussian integers $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$.

Extra Exercises

These **extra** exercises are designed to provide additional challenges for students seeking to deepen their understanding.

These exercises are organised by topic rather than by difficulty.

General.

8.9. Prove that 1 and $p-1$ are the only elements of $Z_p = \mathbb{Z}/p\mathbb{Z}$ that are their own inverses, for any prime p.

Ideals.

8.10. Let R be a ring with unity and I an ideal in R. Show that if some element $a \in I$ has a multiplicative inverse in R, then $I = R$.

8.11. Let R be a ring and define

$$I = \{x \in R \mid x^n = 0, \text{ for some } n \in \mathbb{N}\}.$$

(*1*) Show that if R is commutative, then I is an ideal of R.

(*2*) Provide an example to demonstrate that if R is non-commutative, then I is not necessarily an ideal.

Remark: An element x in a ring R for which there exists an $n \in \mathbb{N}$ such that $x^n = 0$ is called a **nilpotent element**.

Characteristic of a ring.

8.12. Let R be a commutative ring with unity and let $\text{Char}(R) = 3$. Compute and simplify

$$(a + b)^9, \quad a, b \in R.$$

8.13. Let R be a ring in which every element is an idempotent. That is, every $x \in R$ satisfies $x^2 = x$. Show that $\text{Char}(R) = 2$ and that R is commutative.

Field of fractions.

8.14. A **Laurent series** over a field $\mathbb{C}$ is an infinite series of the form:

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^i$$

where $a_i \in \mathbb{C}$ and only finitely many a_i for $i < 0$ are non-zero. This means that while the series can have infinitely many terms with non-negative powers of x, it can only have a finite number of terms with negative powers of x.

(*1*) Prove that the set of all Laurent series $\mathbb{C}[[x, x^{-1}]]$ forms a ring under the usual operations of addition and multiplication of series.

(*2*) Find the field of fractions of $\mathbb{C}[[x, x^{-1}]]$.