ALGEBRAIC STRUCTURES - PRACTICAL 7

This week's Exercises

Solve **all** the exercises of this section (i.e. Exercises 7.1 to 7.3) before the beginning of Semester B.

7.1. Check whether the following are fields.

(*1*) $M = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ a \text{ odd } \right\}$.

(*2*) $N = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ a \text{ even } \right\}$.

(*3*) $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

7.2. Which of the following maps are ring homomorphisms? Determine the kernel of those maps that are homomorphisms.

(*1*) $\theta : \mathbb{Z} \to \mathbb{Z}$ defined by $\theta(a) = 3a$.

(*2*) $\theta : \mathbb{Z}_6 \to \mathbb{Z}_3$ defined by $\theta(\overline{a}) = \overline{a}$.

(*3*) $\theta : \mathbb{C} \to \mathbb{R}$ defined by $\theta(z) = |z|$.

(*4*) $\theta : \mathbb{C} \to \mathbb{C}$ defined by $\theta(z) = iz$.

(*5*) The map $\psi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\psi(f(x)) = f(3)$, for all $f(x) \in \mathbb{Z}[x]$.
(E.g. If $f(x) = x^2 + 1$, then $\psi(f(x)) = f(3) = 3^2 + 1 = 10$.)

7.3. Check whether the following subrings are ideals of the given rings. You do not need to check whether they are subrings.

(*1*) The subset $\mathcal{M}$ of $M(2, \mathbb{R})$ given by

$$\mathcal{M} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

(*2*) The subset of $\mathbb{Z}[x]$ given by

$$\mathcal{N} = \{x \cdot f(x) \mid f(x) \in \mathbb{Z}[x]\}.$$

That is, $\mathcal{N}$ is the set of all polynomials in $\mathbb{Z}[x]$ that are divisible by $g(x) = x$.

7.4. Let $\theta : R \to S$ be a ring homomorphism.

(*1*) Show that the image $\text{Im}(\theta)$ of θ is a subring of S.

(*2*) Show that the kernel $\ker(\theta)$ of θ is a subring of R.

(*3*) Show that $\ker(\theta)$ is an ideal of R.

Recall that

$$\text{Im}(\theta) = \{\theta(r) \mid r \in R\} \subseteq S.$$
$$\text{Ker}(\theta) = \{r \in R \mid \theta(r) = 0_S\} \subseteq R.$$

Practice Exercises

The next exercises are meant to give you some extra practice to better prepare for assessments.

7.5. Let R be a ring and I and J two ideals of it.

(*1*) Show that the intersection $I \cap J$ is also an ideal of R.

(*2*) Show that the sum $I + J$ is also an ideal of R. Recall that

$$I + J = \{i + j \mid i \in I,\, j \in J\}.$$

7.6. Let R be a ring with unity 1_R. Define a map $\psi : \mathbb{Z} \to R$ by the rule

$$\psi(n) = \underbrace{1_R + \cdots + 1_R}_{n \text{ times}}$$

whenever n is positive, and

$$\psi(-n) = \underbrace{-1_R - \cdots - 1_R}_{n \text{ times}}.$$

Also, we set $\psi(0) = 0$.

Show that the map $\psi : \mathbb{Z} \to R$ is a ring homomorphism.

[**Hint:** Remember to consider the cases where m and n are both non-positive, both non-negative, and have different signs. Use the fact that addition in a ring is commutative to narrow down the possibilities.]