ALGEBRAIC STRUCTURES - PRACTICAL 6

This week's Exercises

Solve **all** the exercises of this section (i.e. Exercises 6.1 to 6.4) before Week 21.

6.1. Which elements of $\mathbb{Z}/4\mathbb{Z}$ are zero divisors? Which of $\mathbb{Z}/12\mathbb{Z}$?

6.2. Consider the rings $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$.

(*1*) Which ones have unity? Write down the unities (if any).
(*2*) Which of them have zero divisors? Write down all zero divisors (if any).
(*3*) Which ones are integral domains? Justify.
(*4*) Which ones are fields? Justify.

6.3. Prove that $a^2 - b^2 = (a+b)(a-b)$ for all a, b in a ring R if and only if R is commutative.

6.4.

(*1*) Prove that if n is not prime then $\mathbb{Z}/n\mathbb{Z}$ is not an integral domain.
(*2*) Prove that if p is a prime, then $\mathbb{Z}/p\mathbb{Z}$ is an integral domain.

Practice Exercises

The next exercises are meant to give you some extra practice to better prepare for assessments.

6.5. Let D be an integral domain.

(1) Show that if $a \in D$ satisfies $a^2 = 1$, then a is either 1 or -1.
[*Hint:* Show that $a^2 - 1 = 0$ and factorise the left-hand side.]
(2) Show that if $a \in D$ satisfies $a^2 = a$, then a is either 0 or 1.
(3) Show that if $a \in D$ satisfies $a^n = 0$ for some positive integer n, then $a = 0$.

6.6. Finish the proof of the properties of rings: show that, if $(R, +, \cdot)$ is a ring and $a, b, c \in R$, then

(*1*) Each equation $a + x = b$ (or $x + a = b$) has a unique solution.

(*2*) $-(-a) = a$ and $-(a + b) = (-a) + (-b)$.

(*3*) If m and n are integers, then $(m + n) \cdot a = ma + na$, $m \cdot (a + b) = ma + mb$, and $m(na) = (mn)a$.

In **(3)**, given a positive integer m, what we mean by ma is

$$ma = \underbrace{a + a + \cdots + a}_{m \text{ times}} \quad \text{and } (-m)a = \underbrace{-a - a - \cdots - a}_{m \text{ times}}.$$

6.7. The centre of a ring R is defined to be $\{c \in R \mid cr = rc \text{ for every } r \in R\}$. Show that the centre of a ring **with unity** is a subring.

6.8. What is the smallest subring of $\mathbb{Z}$ containing 3? What is the smallest subring of $\mathbb{R}$ containing $1/2$?

[By smallest we mean with respect to inclusion. For instance, we say that R is the smallest subring of $\mathbb{Z}$ containing 3 if every subring S of $\mathbb{Z}$ containing 3 is such that $R \subseteq S$.]