MTH2004M Differential Equations

Chapter 2 First Order Differential Equations

- Geometrical Interpretation of Solutions
- Separable Differential Equations
- Linear Differential Equations

First Order ODEs

In this Chapter we study **First Order ODEs.** How do we recognise them?

$$\frac{dy}{dx} = f(x, y)$$

- First order –> highest derivative is one
- Ordinary –> differentiation with respect to one *independent variable*
- Unknown *function* $y \equiv y(x)$ to find

Geometric Interpretation of Solutions

Qualitative Analysis of Differential Equations

Differential equations are often very difficult to solve!

We can gain some insight to the solution of a differential equation without having a mathematical expression . . .

We consider the following analysis techniques

- Direction Fields
- Autonomous First-Order ODEs

Direction Fields

By inspecting a first order DE

$$\frac{dy}{dx} = f(x, y)$$

we can derive some properties about the solution e.g.

- How does a solution behave near a certain point?
- How does a solution behave as $x \to \infty$?

To do this we consider the **slope** function . . .

Direction Fields

Definition: The derivative dy/dx of a differentiable function $y \equiv y(x)$ gives the **slopes** of the tangent lines at points on its graph.

The function $f(x, y)$ in the first order ODE

$$\frac{dy}{dx} = f(x, y)$$

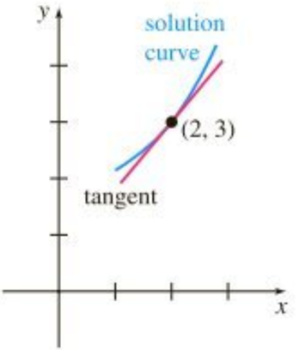
is called the **slope function.**

So the DE itself tells us about the **tangent of the solution curve** at particular points

Example

Consider the first-order ODE

$$\frac{dy}{dx} = 0.2xy \quad \Rightarrow \quad f(x, y) = 0.2xy$$



Consider the point $(2, 3)$.

If a solution passes through (2,3), then the tangent of this solution curve must be

$$f(2, 3) = 0.2 \times 2 \times 3 = 1.2$$

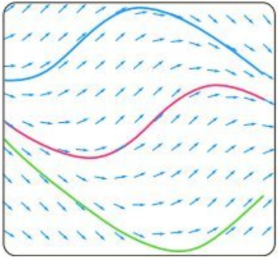
So we already know something about the shape of the solution curve

<u>Definition</u> The **direction field** of a first-order ODE is the collection of line elements defined by slope function $f(x, y)$ evaluated over the xy-plane.

The direction field allows us to visualise what a **family of solution curves** may look like . . .

<u>Example</u>

$$\frac{dy}{dx} = \sin(x + y)$$



Task

Calculate the direction field for the first order ODE

$$\frac{dy}{dx} = y - x$$

for the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$

HINT Fill out the following grid calculating the slope function $f(x, y)$

y \ x	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

Then, sketch the collection of line segments on the *xy*-plane with gradients given by $f(x, y)$

Use a computer....

It’s obviously much more efficient to use a computer to make these sketches!

Check out these sources:

- MATLAB
- Python
- Online tool

Autonomous First Order ODEs

What happens if we have an equation of the form

$$\frac{dy}{dx} = 1 + y^2 = f(\cancel{x}, y)$$

In this equation the slope function DOES NOT depend on the independent variable!

This is called an **autonomous** first order ODE.

Autonomous First Order ODEs

Autonomous first-order ODEs of the form

$$\frac{dy}{dy} = f(y)$$

are common in physical laws that do not change with time e.g.

$$\frac{dT}{dt} = k(T - T_m) \text{ is a temperature model}$$

Instead of plotting direction fields, we can find **critical points** of function f and plot a **phase portrait**

Autonomous First Order ODEs

Definition A **critical** point, c, of an autonomous ODE of the form

$$\frac{dy}{dx} = f(x)$$

is defined by

$$f(c) = 0.$$

Autonomous First Order ODEs

So for autonomous first order ODE

$$\frac{dy}{dx} = f(y)$$

with critical point c satisfying $f(c) = 0$, we can verify that

$$y(x) = c$$

must be a solution to the equation.

Example: Phase Portrait

Consider the autonomous first-order ODE

$$\frac{dP}{dt} = P(a - bP) \equiv f(P)$$

Then the critical points are given by

$$f(P) = P(a - bP) = 0,$$

Example: Phase Portrait

Consider the autonomous first-order ODE

$$\frac{dP}{dt} = P(a - bP) \equiv f(P)$$

Then the critical points are given by

$$\begin{aligned} f(P) = P(a - bP) = 0, \\ \Rightarrow P = 0 \text{ or } P = a/b \end{aligned}$$

We can then divide the *Pt*-plane into a number of regions . . .

Example: Phase Portrait

For critical points

$$f(P) = P(a - bP) = 0,$$
$$\Rightarrow P = 0 \text{ or } P = a/b$$

The critical points define three regions in the *Pt*-plane

1. $(-\infty, 0)$
2. $(0, a/b)$
3. $(a/b, \infty)$

What does $f(P)$ look like in each region? $f(P)$ tells us if $P(t)$ is increasing or decreasing ...

Example: Phase Portrait

For critical points

$$f(P) = P(a - bP) = 0,$$
$$\Rightarrow P = 0 \text{ or } P = a/b$$

The critical points define three regions in the *Pt*-plane

1. $(-\infty, 0)$ –> $f(P)$ is negative
2. $(0, a/b)$ –> $f(P)$ is positive
3. $(a/b, \infty)$ –> $f(P)$ is negative

This allows us to plot our **phase diagram** ...

Example: Phase Portrait

For critical points

$$f(P) = P(a - bP) = 0,$$
$$\Rightarrow P = 0 \text{ or } P = a/b$$

The critical points define three regions in the *Pt*-plane

1. $(-\infty, 0)$ –> $f(P)$ is negative –> P is decreasing
2. $(0, a/b)$ –> $f(P)$ is positive –> P is increasing
3. $(a/b, \infty)$ –> $f(P)$ is negative –> P is decreasing

This allows us to plot our **phase diagram** ...

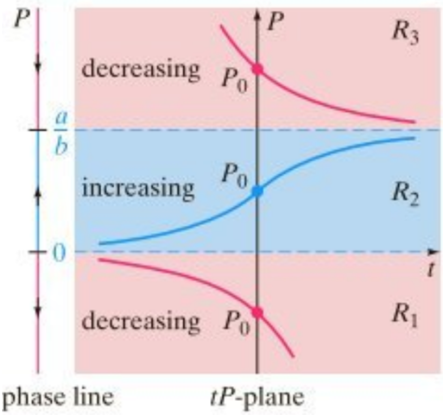
Example: Phase Portrait

The autonomous first-order ODE

$$\frac{dP}{dt} = P(a - bP) \equiv f(P)$$

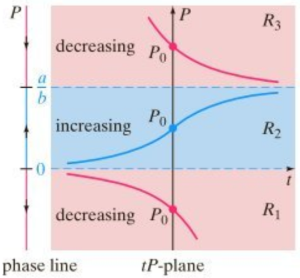
has **phase diagram**:

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	points down
$(0, a/b)$	plus	increasing	points up
$(a/b, \infty)$	minus	decreasing	points down



Phase portraits

Phase portraits enable us to determine the **stability** of the solution by **classifying** the critical points



- $P = 0$ is asymptotically unstable; a **repeller** or a "source"
- $P = a/b$ is asymptotically stable; an **attractor** or a "sink"

NOTE asymptotically semi-stable critical points are called "nodes"

Task: Classifying Critical Points

Find and classify the critical points of the following autonomous first-order ODE by sketching the phase diagram

$$\frac{dP}{dt} = \left(1 - \frac{P}{20}\right)^3 \left(\frac{P}{5} - 1\right) P^7$$