MTH2003 - Complex Analysis

Basics:

Form: $z = x + iy : x, y \in \mathbb{R} \ \cap \ i = \sqrt{-1}$

- $\text{Re}(z) = x,$
  $\text{Im}(z) = y.$

Vector Space: $C = \{x + iy : x, y \in \mathbb{R}\}$

Magnitude: $|z| = \sqrt{x^2 + y^2}$

Complex Conjugate: $\bar{z} = x - iy$

- $z + \bar{z} = 2\text{Re}(x)$
- $z - \bar{z} = 2i\text{Im}(z)$

Properties:

- Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z| = \sqrt{z \cdot \bar{z}}$
- $z_1 \bar{\cdot} z_2 = \bar{z_1} \cdot \bar{z_2} \ \cap \ z_1 \bar{+} z_2 = \bar{z_1} + \bar{z_2}$
- If $z = \bar{z} \implies z \in \mathbb{R}$

Polar Coordinates:

- $x = \text{Re}(z) = r\cos\theta$
- $y = \text{Im}(z) = r\sin\theta$
  - $r = |z|$
  - $\arg(z) = \theta : \tan\theta = \frac{y}{x}$

Exponential Form: $z = re^{i\theta}$

- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
- Euler's Identity: $e^{i\pi} + 1 = 0$

Solving Complex Algebraic Equations:

- For an equation $z^a = x + iy$, find the exponential form of the $\text{LHS} = re^{i\theta} : \theta \in [0, 2\pi)$

- In general $x + iy = e^{i(\theta + 2\pi m)} : m \in \mathbb{Z}$
- $m = \{0, 1, ...a - 1\}$
- $z = e^{i\frac{(\theta+2\pi m)}{a}} : m \in \mathbb{Z}$

DeMoirve: $z^n = r^n(\cos n\theta + i \sin n\theta)$

- Roots of Unity:
    - For $z^n = 1,$
        - $z^n = r^n(\cos n\theta + i \sin n\theta)$
        - $1 = 1 + 0 \cdot i = \cos 2k\pi + \sin 2k\pi : k = 0, 1, 2, ...$
    - $\begin{cases} r^n \cos n\theta = \cos 2k\pi \\ r^n \sin n\theta = \sin 2k\pi \end{cases}$
    - By inspection $r^n = 1 \ \cap \ \theta = \frac{2k\pi}{n}$
    - So $z = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} : k = 0, 1, ..., n - 1$ are the n roots of unity.
        - $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ the roots of unity of $z^n = 1$: $\{1, \omega, \omega^2, ..., \omega^{n-1}\}$

Topology of the Complex Plane:

Function of a Complex Variable:

Multi-valued Functions: