Algebraic Structures - Practical 1

This week's mandatory exercises

Solve **all** the exercises of this section (1.1 to 1.3) before the lecture of Week 3.

Exercise 1 is very important, as you should get comfortable with showing that a set is a group with a given operation, because this will be used the whole year.

1.1. Which of the following are groups? If they are groups, check closure and associativity and give the identity element and the inverse of each element. If they are not, say why.

(a) $\{1\}$ with multiplication;

(b) $\mathbb{C}$ with addition;

(c) $\mathbb{C}$ with multiplication;

(d) $\{1, -1\}$ with multiplication;

(e) $\{-1, 0, 1\}$ with addition;

(f) $\mathbb{Z}$ with operation defined by $a \odot b = a + b - 3$;

(g) $\mathbb{R}$ with operation defined by $a \otimes b = a + b - ab$;

(h) The set $\mathbb{Z}[x]$ of polynomials with integer coefficients and multiplication. [That is, $\mathbb{Z}[x]$ is the set of all polyinomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with n a positive integer and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$.]

(i) The set $\mathbb{Q}[x]$ of polynomials with rational coefficients with addition; [That is, $\mathbb{Q}[x]$ is the set of all polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with n a positive integer and $a_0, a_1, \ldots, a_n \in \mathbb{Q}$.]

1.2. Show that if G is a group operation denoted multiplicatively, then

$$(gh)^{-1} = h^{-1} g^{-1}.$$

1.3. Consider the set of diagonal matrices

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}^* \right\}.$$

(a) Show that D with usual matrix multiplication is a group.

(b) Is this group abelian? Justify your answer.

Practice Exercises

The next exercises are meant to give you some extra practice to better prepare for assessments.

1.4. Give an example of two 2×2 real matrices A and B such that $AB \neq BA$. Justify your answer.

1.5. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 3 & -5 \end{pmatrix}$.

a) Compute AB.
b) Compute A^{-1} and B^{-1}.
c) Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

1.6. Let F denote the set of all mappings from $\mathbb{R}$ to $\mathbb{R}$. The operation on F is defined as follows: For $f, g \in F$ define $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in \mathbb{R}$.

(a) For $f, g \in F$ check that $f + g \in F$.

(b) Show that F is a group with this operation.

1.7. Let A denote the set of functions $\alpha_{a,b} : \mathbb{R} \to \mathbb{R}$ with $a \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ defined by

$$\alpha_{a,b}(x) = ax + b, \text{ for each } x \in \mathbb{R}.$$

Define the operation $\circ$ to be composition of functions (that is $f \circ g(x) = f(g(x))$).

(a) Show that A with $\circ$ is a group.

(b) Is this group abelian?